Math 279 Lecture 13 Notes

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1 Making the Jump From Stochastic ODEs to PDEs

1.1 Main results thus far for solving stochastic ODEs

Here are two main results that we have established so far:

1. The "ODE"
$$\begin{cases} \dot{y} = \sigma(y)\dot{x} \\ y(0) = y^0 \end{cases}$$
has a solution that is stable with respect to its input, provided we use the rough-path interpretation for the integrals:
$$y(t) = y^0 + \int_0^t (\sigma(y), \widehat{\sigma})\, d(x, \mathbb{X})$$
with $\widehat{\sigma} = D\sigma(y)\sigma(y)$.

   Moreover, y is a fixed point of the operator
$$\mathscr{I}(\mathbf{y}, \mathbf{x}) = y^0 + \int_0^t (\sigma(y) D\sigma(y)\widehat{y})\, d\mathbf{x}.$$
Using our bounds for the integral, the operator $\mathscr{I}$ is bounded linear in $\mathbf{x}$ and locally Lipschitz in $\mathbf{y}$, and we learn that the solution $X(y^0, \mathbf{x})$ is continuous.

2. If B denotes the standard Brownian motion, then we have two rather natural candidates for its (random) lift, namely $(B, \mathbb{B})$ (Itô) and $(B, \widehat{\mathbb{B}})$ (Stratonovich) in $\mathscr{R}^\alpha$ for any $\alpha \in (0, 1/2)$. Note that our candidate $(B(\cdot), \mathbb{B}(\cdot, \cdot\,; B))$ is in $L^2(\mathbb{P})$ with $\mathbb{P}$ representing the Wiener measure, though $\mathbb{B}$ as a function of B is only measurable.

   In particular, we may approximate B by some nice function, say $B^{(n)}$, and solve
$$\begin{cases} \dot{y} = \sigma(y)\dot{B}^{(n)} \\ y(0) = y^0. \end{cases}$$

Then $\lim_{n\to\infty} y_n = y$, where y solves

$$\dot{y} = \sigma(y)\dot{\widehat{\mathbb{B}}}.$$

Indeed, if for $B^{(n)}$, we choose the linear interpolation of B using dyadic points $D_n = \{i/2^n : i \in \mathbb{Z}\}$ and consider $(B^{(n)}, \mathbb{B}^{(n)})$ by

$$\widehat{\mathbb{B}}^{(n)}(s,t) = \int_s^t B^{(n)} \otimes \dot{B}^{(n)}(\theta)\, d\theta,$$

then as we discussed last time, $\mathbb{B}^{(n)}(s,t)$ is simply the Stratonovich approximation. Hence, in the L^2 sense, $\mathbb{B}^{(n)} \to \widehat{\mathbb{B}}$.

We also know that $\sup_n \|[B^{(n)}, \widehat{\mathbb{B}}^{(n)}]_{\alpha,2\alpha}\|_{L^q(\mathbb{P})} < \infty$. As a result, if we define $\mathbf{B}^{(n)} = (B^{(n)}, \widehat{\mathbb{B}}^{(n)})$ and $\widehat{\mathbf{B}} = (B, \widehat{\mathbb{B}})$ and regard it as a function B, we can show that for $\mathbb{P}$-almost all choices of B,

$$d_\alpha(\mathbf{B}^{(n)}, \widehat{\mathbb{B}}) \to 0,$$

where d_α is the distance with respect to $[\cdot]_{\alpha,2\alpha}$.

In summary, we managed to do Stochastic calculus in two steps:

$$B \xrightarrow{\text{measurable}} (B, \mathbb{C}) \xrightarrow{\text{continuous}} \text{“}\dot{y} = \sigma(y)\tfrac{d}{dt}(B, \mathbb{B})\text{”}$$

Now we want to carry out the program for PDEs.

1.2 Preliminaries for Stochastic PDEs

We start with some notation. We have $\varphi : \mathbb{R}^d \to \mathbb{R}$ or $\varphi : D \to \mathbb{R}$ with some open subset $D \subseteq \mathbb{R}^d$. We will use

$$\|\varphi\|_{L^\infty} = \|\varphi\|_\infty = \sup_x |\varphi(x)|, \qquad \|\varphi\|_{L^\infty(D)} = \sup_{x\in D} |\varphi(x)|$$

to denote the L^∞ norm on $\mathbb{R}^d$ and D, respectively. Given $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$, we define

$$\partial^k(\varphi) = \partial_{x_d}^{k_d} \cdots \partial_{x_1}^{k_1}\varphi, \qquad |k| = k_1 + \cdots + k_d.$$

We write C^r for the set of functions φ for which ∂^k exists and is continuous for any k with $|k| \le r$. And

$$\|\varphi\|_{C^r} = \sum_{|k|\le r} \|\partial^k\varphi\|_{L^\infty}.$$

We write $\mathcal{D}$ for the set of smooth functions of compact support, and if K is a compact subset of $\mathbb{R}^d$, then $\mathcal{D}(K)$ means the set of $\varphi \in \mathcal{D}$ with $\operatorname{supp}\varphi \subseteq K$. By $\mathcal{D}'$, we mean the set of linear functionals $T : \mathcal{D} \to \mathbb{R}$ which are linear and satisfy

$$|T(\varphi)| \le c_K \|\varphi\|_{C^r K}$$

for some constant c_K and index r_K for every $\varphi \in \mathcal{D}(K)$. Here, r_K is called the **order** of the distribution.

Example 1.1. A 0-th order distribution would be a measure by the Riesz representation theorem.

Next, we wish to discuss $\mathcal{C}^\alpha(\mathbb{R}^d)$ (or $\mathcal{C}^\alpha_{\text{loc}}(\mathbb{R}^d)$) for $\alpha \in \mathbb{R}$. Given a (test) function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we define

$$\varphi_a^\delta(x) = \delta^{-d}\varphi\left(\frac{x-a}{\delta}\right), \qquad (\varphi^\delta := \varphi_0^\delta, \varphi_a := \varphi_a^\delta).$$

Observe that $\int \varphi_a^\delta = \int \varphi$.

Imagine that $u : \mathbb{R}^d \to \mathbb{R}$ is Hölder of exponent α, and take φ from

$$\mathcal{D}_0 = \left\{\varphi \in \mathcal{D} : \operatorname{supp}\varphi \subseteq B(0,1), \int \varphi \neq 0, \|\varphi\|_{L^\infty} \leq 1\right\}.$$

We will use the bracket notation

$$\langle u - u(a), \varphi_a^\delta\rangle = \int (u - u(a))\varphi_a^\delta \, dx.$$

Taking absolute values and making a change of variables, we can write

$$\begin{aligned} |\langle u - u(a), \varphi_a^\delta\rangle| &= \left|\int (u - u(a))\varphi_a^\delta \, dx\right| \\ &= \left|\int (u(a + \delta z) - u(a))\varphi(z) \, dz\right| \\ &\leq [u]_\alpha \delta^\alpha \int |z| \cdot |\varphi(z)| \, dz. \end{aligned}$$

Hence, for $u \in \mathcal{C}^\alpha$ with $\alpha \in (0,1]$,

$$[\![u]\!]_{\mathcal{C}^\alpha} := \sup_{\delta \in (0,1]} \sup_{a \in K} \sup_{\varphi} \frac{|\langle u - u(a), \varphi_a^\delta\rangle|}{\delta^\alpha} \leq c[u]_\alpha,$$

so these norms are equivalent by the following proposition:

Proposition 1.1. *If $[\![u]\!]_{\mathcal{C}^\alpha} < \infty$, then $u \in \mathcal{C}^\alpha$.*

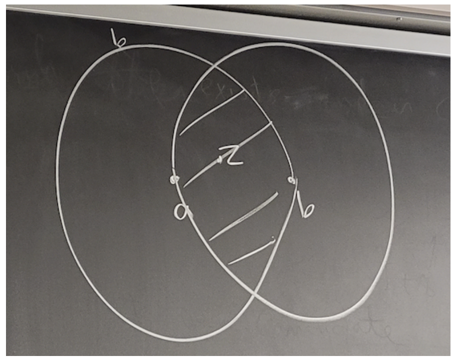
Proof. If $[\![u]\!]_{\mathcal{C}^\alpha} < \infty$,

$$\sup_{a \in K} \delta^{-d} \int_{|z-a|<\delta} |u(z) - u(a)| \, dz \leq c_0 \delta^\alpha.$$

Choose $\delta = |a-b|$ and argue that

$$|u(a) - u(b)| \le |u(a) - u(z)| + |u(z) - u(b)|$$

for $z \in B(a,\delta) \cap B(b,\delta)$ with $\delta = |a-b|$.



Integrate both sides over $B(a,\delta) \cap B(b,\delta)$ to get

$$\begin{aligned}
|\underbrace{B(a,\delta) \cap B(b,\delta)}_{B_{a,b}}| \cdot |u(a) - u(b)| &\le \int_{B_{a,b}} |u(a) - u(z)|\, dz + \int_{B_{a,b}} |u(b) - u(z)|\, dz \\
&\le \int_{B(a,\delta)} |u(a) - u(z)|\, dz + \int_{B(b,\delta)} |u(b) - u(z)|\, dz \\
&\le 2c_0 \delta^{\alpha+d}.
\end{aligned}$$

Hence, $|u(a) - u(b)| \le c_1 \delta^\alpha$, as desired. $\square$

We want to go beyond $\alpha \in (0,1)$. For example, consider $\alpha \ge 1$. For such α, we first define $n = \max\{m \in \mathbb{N} : m < \alpha\}$. We say $u \in \mathcal{C}^\alpha$ if u has n-many derivatives and if

$$P_a^u(x) := \sum_{|k| \le n} (\partial^k u)(a)(x-a)^k, \qquad (x-a)^k := \prod_{i=1}^d (x_i - a_i)^{k_i}, k! := k_1! \cdots k_d!,$$

then

$$[\![u]\!]_{\alpha,K} = \sup_{\delta \in (0,1)} \sup_{\varphi \in \mathcal{D}_0} \sup_{a \in K} \frac{\int (u - P_a^u) \varphi_a^\delta \, dx}{\delta^\alpha} < \infty.$$

One can show that $[\![u]\!]_{\alpha,K} < \infty$ if and only if u possesses n many derivatives and for any k with $|k| = n$, $\partial^k u$ is Hölder of exponent $\alpha - n$.

Basically, we need to choose $\varphi = \partial^k \psi$ for some smooth ψ, and observe that

$$\|\partial^k \psi\|_{L^\infty} \le \lambda^{-k-d}.$$